Math 249, Monday, Mar. 30

Frobenius characteristic $\quad F : X(S_n) \to (\Lambda)_n$

$$F\chi = \frac{1}{n!} \sum_{w \in S_n} \chi(w)\, p_{\tau(w)} \qquad \tau(w) = \text{cycle type of } w$$

$$= \sum_{\lambda} \chi(w_\lambda)\, p_\lambda / z_\lambda \qquad \#w : \tau(w) = \lambda \text{ is } \frac{n!}{z_\lambda}$$

i.e. $\langle F\chi, p_\lambda \rangle = \chi(w_\lambda)$.

Key computation: $F\, \mathrm{Ind}_{S_\lambda}^{S_n}(1)$ $\qquad S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k} \subset S_n$

S_{λ_1} acts on $\{1, \dots, \lambda_1\}$, S_{λ_2} acts on $\{\lambda_1+1, \dots, \lambda_1+\lambda_2\}$

Digression

Restriction and induction

$H \subset G$ subgroup

Restriction: given $G \curvearrowright V$, then $H \curvearrowright V$, call it $V|H$

$$\chi_{V|H}(h) = \chi_V(h) \qquad h \in H.$$

Induction: given $H \curvearrowright V$ $\qquad V$ is a module for $\mathbb{C}H$

$$\mathrm{Ind}_H^G(V) = \mathbb{C}G \otimes_{\mathbb{C}H} V \qquad gh \otimes v = g \otimes hv$$

More explicitly: let $g_1H, \dots, g_kH$ be all distinct cosets of H in G. Let $v_1, \dots, v_d$ be a $\mathbb{C}$ basis of V: then $\{g_i \otimes v_k\}$ form a $\mathbb{C}$-basis of $\mathrm{Ind}_H^G(V)$, g acts by

$$g(g_i \otimes v_k) = gg_i \otimes v_k = g_j h \otimes v_k = g_j \otimes \underbrace{hv_k}_{\in V}$$

$gg_i \in g_jH$ for some j

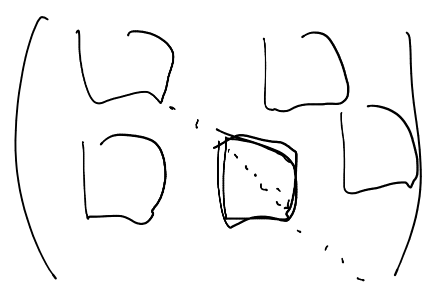
$gg_i = g_j h$, $h \in H$

$\chi_{\mathrm{Ind}_H^G(V)}(g) = ?$ trace only involves diagonal blocks:

$$gg_i \in g_iH$$
$$g_i^{-1}gg_i \in H$$

$$g(g_i \otimes v_k) = g_i \otimes (\underbrace{g_i^{-1}gg_i}_{h} v_k)$$

trace is sum of $\chi_V(g_i^{-1}gg_i)$ over i s.t. $g_i^{-1}gg_i \in H$



algebra A

subalgebra $B \subset A$

left B module V

A is left A module, right B module

$\to A \otimes_B V$ $\quad ab \otimes v = a \otimes bv$

$$(\mathrm{Ind}_H^G \chi)(g) = \sum_i \tilde{\chi}(g_i^{-1} g g_i) \qquad \tilde{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{otherwise} \end{cases}$$

Notice $g_i' = g_i h \Rightarrow \tilde{\chi}(g_i'^{-1} g g_i') = \tilde{\chi}(h^{-1} g_i^{-1} g g_i h) = \tilde{\chi}(g_i^{-1} g g_i)$

$$(\mathrm{Ind}_H^G \chi)(g) = \frac{1}{|H|} \sum_{w \in G} \tilde{\chi}(w^{-1} g w)$$

Ex. $\mathrm{Ind}_H^G(\mathbb{C}) = \mathbb{C}(G/H)$ ← trivial rep of H

$\mathrm{Ind}_H^G(1_H)(g) = |(G/H)^g|$

$g_i \otimes 1$

$g_i H$

$g g_i \otimes 1 = g_j \otimes h \cdot 1$

$g g_i H = g_j H$

$g(g_i \otimes 1) = (g_i \otimes 1)$

Frobenius Reciprocity:

$$\langle \chi, \mathrm{Ind}_H^G \varphi \rangle_G = \langle \chi|_H^G, \varphi \rangle_H$$

χ is a G char., φ is an H char.

Proof. $\mathrm{LHS} = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) (\mathrm{Ind}_H^G \varphi)(g)$

$$= \frac{1}{|G||H|} \sum_{g, w \in G} \chi(g^{-1}) \tilde{\varphi}(w^{-1} g w)$$

$h = w^{-1} g w$

$g = w h w^{-1}$

$$= \frac{1}{|G|\,|H|} \sum_{h,w \in G} \chi(w h^{-1} w^{-1})\, \tilde{\varphi}(h)$$

$$= \frac{1}{|G|} \frac{1}{|H|} \sum_{\substack{h \in H \\ w \in G}} \chi(h^{-1})\, \varphi(h) = \frac{1}{|H|} \sum_{h \in H} \chi(h^{-1})\, \varphi(h)$$

$$= RHS.$$

$B \subset A$, $B \curvearrowright W$, $A \curvearrowright V$:

$$\mathrm{Hom}_A(\mathrm{Ind}_B^A W, V) \cong \mathrm{Hom}_B(W, V|_B)$$

$(a \otimes w \mapsto a \cdot \alpha(w)) \longleftarrow\!\!\shortmid \; \alpha : W \to V$

$\mathrm{Ind}_B^A(-)$ is left adjoint to $(-)|_B^A$

$\Rightarrow$ Frobenius Reciprocity.

Fx for $\chi = \mathrm{Ind}_{S_\lambda}^{S_n}(1)$ = char of $\mathbb{C}\cdot(S_n/S_\lambda)$

Word $\underbrace{a_1, \dots, a_1}_{\lambda_1}, \underbrace{a_2 \cdots a_2}_{\lambda_2}, \dots, \underbrace{a_k, \dots, a_k}_{\lambda_k}$ length $= n$

$|\lambda| = n$

$\lambda = (3,3,2)$ $\quad$ $\underbrace{aaa}\,\underbrace{bbb}\,\underbrace{cc}$ $\quad$ $n = 8$

Stabilizer of λ is S_λ.

$S_n/S_\lambda \cong \{$words with λ_1 a_1's, λ_2 a_2's, ...$\}$

Consider species $I(S) = \{\text{maps } f: S \to \mathbb{N}\}$ weighted by $\prod_{s\in S} a_{f(s)}$.

$S = [8]$ $\quad 1\,2\,1\,4\,2\,1\,3\,2 : [8] \to \mathbb{N}$

weight $a_1^3 a_2^3 a_3 a_4 = a^\lambda$

$I(S)$ is product of trivials species for each $i \in \mathbb{N}$

Cycle index for trivial species is $\Omega = \exp \sum_{k\geq 1} p_k/k = \sum_\lambda p_\lambda / z_\lambda = \sum_n h_n$

With weight a_i^n on set of size n:

$$\Omega[a_i X] = \exp \sum a_i^k p_k/k = \sum_n a_i^n h_n$$

Product over all i: $\quad \Omega[a_1 X]\,\Omega[a_2 X] \cdots$

$$= \Omega[(a_1 + a_2 + \cdots) X] = \Omega[AX] \quad A = a_1 + a_2 + \cdots$$

$$= \sum m_\lambda(A)\, h_\lambda(X)$$

$$Z_I = \sum_n \frac{1}{n!} \sum_{w\in S_n} |I([n])^w|\, p_{\tau(w)}$$

↖ OGF in $a_1, a_2, \ldots$ for weighted count

Take coefficient of a^λ in (↑): $\quad |I([n])^w| \longmapsto (S_n/S_\lambda)^w = \mathrm{Ind}_{S_\lambda}^{S_n}(1)\,(w)$

Coef. of a^λ in Z_I is $\dfrac{1}{n!} \sum_{w\in S_n} (\mathrm{Ind}_{S_\lambda}^{S_n} 1)(w)\, p_{\tau(w)}$

$n = |\lambda|$ ↑

$$= F(\mathrm{Ind}_{S_\lambda}^{S_n} 1) = h_\lambda$$

$= \Omega[AX]$

$= \sum m_\lambda(A)\, h_\lambda(X)$

$$Z_G = \sum_n \underbrace{\frac{1}{n!} \sum_{w \in S_n} |G([n])^w| \, p_{\tau(w)}}_{= F_{\mathbb{C}G([n])}}$$

$$Z_G = \sum_n{}' F_{\mathbb{C}G([n])}$$

Ex. $\mathbb{C}G = \mathrm{Ind}_1^G 1 \Rightarrow F_{\mathbb{C}S_n} = F(\mathrm{Ind}_{S_{1,1,\dots,1}}^{S_n} 1) = h_{1^n} = h_1^n$

Prop. $F : X(S_n) \to (\Lambda)_n$ is an isometry.

Proof: $\langle F\chi, F\varphi \rangle = \langle \frac{1}{n!} \sum_v \chi(v) p_{\tau(v)}, \frac{1}{n!} \sum_w \varphi(w) p_{\tau(w)} \rangle$

$$= \langle \sum_\lambda \chi(w_\lambda) p_\lambda / z_\lambda, \sum_\mu \varphi(w_\mu) p_\mu / z_\mu \rangle$$

$$= \frac{1}{n!} \sum_\lambda \chi(w_\lambda) \varphi(w_\lambda^{-1}) \frac{n!}{z_\lambda}$$

$$\langle p_\lambda, p_\lambda \rangle = z_\lambda$$

$$\langle p_\lambda / z_\lambda, p_\lambda / z_\lambda \rangle = \frac{1}{z_\lambda}$$

$$= \frac{1}{n!} \sum_{w \in S_n} \chi(w) \varphi(w^{-1}) \qquad \frac{n!}{z_\lambda}$$

$$= \langle \chi, \varphi \rangle \quad \blacksquare$$

Since $s_\lambda \in \Lambda_{\mathbb{Z}} = \mathbb{Z} \cdot \{h_\mu\}$ $\qquad h_\mu = F(\text{a character})$

$\Rightarrow s_\lambda = F(\text{a virtual character over } \mathbb{Z}) = F(\chi_\lambda)$ ← Def.

$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu} \Rightarrow \langle \chi_\lambda, \chi_\mu \rangle = \delta_{\lambda\mu}$.

$\langle \chi_\lambda, \chi_\lambda \rangle = 1 = \sum c_i^2$ $\qquad \chi_\lambda = \sum c_i \chi_i \quad c_i \in \mathbb{Z}$, χ_i irr.

One $c_i = \pm 1$, others 0. $\qquad \langle \chi_\lambda, \chi_\lambda \rangle = \sum c_i^2$

$\chi_\lambda = \pm \chi_i$

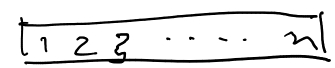
$\langle s_\lambda, p_1^n \rangle = \langle F\chi_\lambda, p_1^n \rangle = \chi_\lambda(1) = \pm \dim V_i$

$\langle s_\lambda, h_{1^n} \rangle = \langle m_{1^n} \rangle s_\lambda = |SYT(\lambda)|$ = # standard Young tableaux

$\Rightarrow$
- The $\chi_\lambda = F^{-1} s_\lambda$ are the irr. chars of S_n.
- Their dimensions are $\chi_\lambda(1) = |SYT(\lambda)|$
- The character table is given by $\chi_\lambda(w_\tau) = \langle s_\lambda, p_\tau \rangle$

Ex. $h_n = \sum_\lambda p_\lambda / z_\lambda \qquad \left(\Omega = \exp \sum_{k \geq 0} p_k / k \right)$

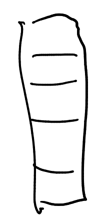
$\langle h_n, p_\lambda \rangle = 1 \qquad h_n = s_{(n)}$



$\chi_{(n)}(w) = 1 \quad \forall w$ is the trivial char

$e_n = \sum_\lambda (-1)^{n-\ell(\lambda)} p_\lambda / z_\lambda \qquad (-1)^{n-\ell(\lambda)} = \varepsilon(w_\lambda)$

$\langle e_n, p_\lambda \rangle = \varepsilon(w_\lambda) \qquad s_{(1^n)} = e_n \qquad \chi_{(1^n)}(w) = \varepsilon(w_\lambda)$



$\omega p_\lambda = (-1)^{n-\ell(\lambda)} p_\lambda \qquad \omega s_\lambda = s_{\lambda^*}$

$$\varepsilon \otimes \chi_\lambda \longleftrightarrow \omega F \chi_\lambda = \omega s_\lambda = s_{\lambda^*} = F\chi_{\lambda^*} \longleftrightarrow \chi_{\lambda^*}$$

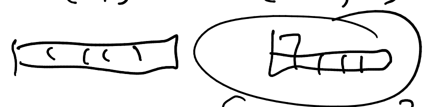
$$\varepsilon \otimes \chi_\lambda = \chi_{\lambda^*}$$

$1\ 1\ 1 \cdots 1\ 1\ 2$

Defining rep. $S_n \curvearrowright \mathbb{C}^n$ is $\mathbb{C}\cdot\{\text{words of weight } 1^{n-1}, 2\}$

$$= \mathbb{C}\, S_n / S_{n-1,1} = \mathrm{Ind}_{S_{n-1,1}}^{S_n} 1$$

$$F\chi_{\mathbb{C}^n} = h_{n-1,1}$$
$$= h_{n-1} h_1 =$$
$$= s_{(n)} + s_{(n-1,1)}$$



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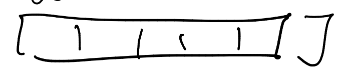

$\mathbb{C}^n \cong \mathbb{C} \oplus V \qquad V = \{ \underline{v} \in \mathbb{C}^n \mid \sum v_i = 0 \} \qquad V \cong \mathbb{C}^n / \mathbb{C}\cdot(1, \dots, 1)$

$\chi_V = s_{(n-1,1)} \Rightarrow V$ is irreducible.